

New Properties of Higher-Order Radial Sets and Higher-Order Radial Derivatives and Applications to Optimality Conditions¹

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Abstract: Several new properties are obtained for higher-order upper radial sets and corresponding derivatives introduced in [J. Glob. Optim. DOI 10.1007/s10898-012-9861-z]. Higher-order sufficient optimality conditions and higher-order necessary optimality conditions are established for weak efficient solutions of set-valued optimization problems. Several corollaries are provided to show that obtained results include a few existing ones. In addition, one removes deficiencies contained of two results in [J. Glob. Optim. DOI 10.1007/s10898-012-9861-z].

Keywords: Higher-order upper radial sets, Higher-order upper radial derivatives, Set-valued vector optimization, Weak efficiency, Higher-order optimality conditions.

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1 Introduction

In set-valued analysis, the notion of a derivative of a set-valued map has been formulated in different ways and applied to set up the optimality conditions (see [1–11]). By virtue of the concept of contingent derivative for a set-valued map (see [2]), Corley [6] investigated optimality conditions for set-valued optimization problems. But it turns out that necessary and sufficient optimality conditions do not coincide under standard assumptions. For solving this problem, Jahn and Rauh [7] introduced the contingent epiderivative of a set-valued map and then obtained unified necessary and sufficient optimality conditions. But, unfortunately, since the contingent epiderivative of a set-valued map is a single-valued map, the conditions assuring the existence of the contingent epiderivative are hard to be satisfied. To overcome the difficulty, Chen and Jahn [8] introduced a generalized contingent epiderivative of a set-valued map, and established a unified necessary and sufficient optimality condition for set-valued optimization problems in terms of the generalized contingent epiderivative. Li and Chen [9] proposed higher-order generalized contingent (adjacent) epiderivatives of set-valued maps, and then obtained higher-order Fritz John type necessary and sufficient conditions for Henig efficient solutions to a constrained set-valued optimization problem. Li et al. [10] studied some properties of higher-order tangent sets and higher-order derivatives introduced in [2], and then obtained higher-order necessary and sufficient optimality conditions for set-valued optimization problems under cone-concavity assumptions. In [11], Li et al. introduced generalized second-order composed contingent epiderivatives for set-valued maps and established a unified sufficient and necessary optimality condition for set-valued optimization problems by employing the generalized second-order composed contingent epiderivatives. In [12], using the concept of the radial epiderivatives, Kasimbeyli obtained necessary and sufficient optimality conditions for optimization problems without convexity conditions. By employing higher-order upper radial set and higher-order upper radial derivative, Anh et al. [13] established optimality conditions of weakly efficient solutions for set-valued optimization problems. Wang et al. [14] proposed the higher-order weak radial epiderivative of a set-valued map, and obtained the optimality conditions for non-convex set-valued optimization problems under the weakly efficiency. Zhang and Wang [15] introduced the second-order weakly composed radial epiderivative of set-valued maps, and obtained the necessary optimality conditions of Benson proper efficient solutions for the constrained set-valued optimization problems without the assumptions of generalized cone-convexity. Peng et al. [16] provided

the higher-order weak lower inner Studniarski epiderivative for set-valued maps, and obtained KarushCKuhnCTucker necessary optimality conditions for Benson proper efficient solutions of the constrained set-valued optimization problems.

Motivated by the work reported in [10–16], several new properties are obtained for higher-order upper radial sets and higher-order upper radial derivatives introduced in [13], and by virtue of the properties, a few optimality conditions are established for weak efficient solutions of a set-valued optimization problem. Some recent existing results are derived from the obtained ones. In addition, one removes deficiencies contained in two earlier results in [13].

The rest of the paper is organized as follows. In Section 2, we recall some basic concepts. In Section 3, we obtain several properties of the higher-order upper radial sets and higher-order upper radial derivatives. In Section 4, we establish optimality conditions for weak efficient solutions of constrained set-valued optimization problems.

2 Preliminaries

Throughout this paper, if not otherwise specified, let X, Y and Z be three real normed spaces, where the spaces Y and Z are partially ordered by nontrivial pointed closed convex cones $C \subset Y$ and $D \subset Z$ with $\text{int}C \neq \emptyset$ and $\text{int}D \neq \emptyset$, respectively. One assumes that $0_X, 0_Y, 0_Z$ denote the origins of X, Y, Z , respectively, Y^* denotes the topological dual space of Y and C^* denotes the dual cone of C , defined by $C^* = \{\varphi \in Y^* | \varphi(y) \geq 0, \forall y \in C\}$. Let S be a nonempty subset of X , $F : S \rightarrow 2^Y$ and $G : S \rightarrow 2^Z$ be two given nonempty set-valued maps. The graph of F is defined by $\text{gr}F = \{(x, y) \in X \times Y | x \in S, y \in F(x)\}$. The profile map $F_+ : S \rightarrow 2^Y$ is defined by $F_+(x) = F(x) + C$, for every $x \in S$.

Definition 2.1 [2] *Let M be a nonempty subset in X , $\check{x} \in M$. We say that \check{x} is a C -weakly efficient point of M if*

$$(M - \{\check{x}\}) \cap (-\text{int}C) = \emptyset.$$

Definition 2.2 (See [17]) *Let $S \subseteq X$ be a nonempty subset and $x_0 \in \text{cl}S$. The closed radial cone $T_S^r(x_0)$ to S at x_0 is the set of all $v \in X$ for which there exist a sequence $\{\lambda_n\}$ of positive real numbers and a sequence $\{x_n\}$ in S with $\lim_{n \rightarrow \infty} x_n = v$ such that $x_0 + \lambda_n x_n \in S$, for all $n \in N$, where N denotes the natural number set.*

Definition 2.3 (See [17]) Let $S \subseteq X$ be a nonempty subset, $F : S \rightarrow Y$ be a set-valued map and $(x_0, y_0) \in \text{gr}F$. The radial derivative $D^r F(x_0, y_0)$ of F at (x_0, y_0) is a set-valued map from X to Y defined by

$$y \in D^r F(x_0, y_0)(x) \text{ if and only if } (x, y) \in T_{\text{gr}F}^r(x_0, y_0).$$

Definition 2.4 (See [13]) Let $x \in S \subseteq X$, $F : S \rightarrow 2^Y$, $(x_0, y_0) \in \text{gr}F$ and $(u_1, v_1), \dots, (u_{m-1}, v_{m-1}) \in X \times Y$ with $m \geq 1$.

(i) The m th-order upper radial set of S with respect to u_1, \dots, u_{m-1} is defined as

$$\begin{aligned} & T_S^{r(m)}(x, u_1, \dots, u_{m-1}) \\ &= \{x \in X \mid \exists t_n > 0, \exists x_n \rightarrow x, \forall n, x_0 + t_n u_1 + \dots + t_n^{m-1} u_{m-1} + t_n^m x_n \in S\}. \end{aligned}$$

(ii) The m th-order upper radial derivative of F at (x_0, y_0) with respect to $(u_1, v_1), \dots, (u_{m-1}, v_{m-1})$ is the set-valued map $D_R^{(m)} F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1}) : X \rightarrow 2^Y$ whose graph is

$$\text{gr}D_R^{(m)} F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1}) = T_{\text{gr}F}^{r(m)}(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1}).$$

From the definition, one knows that the following results hold.

Proposition 2.1 Let $x \in S \subseteq X$, $F : S \rightarrow 2^Y$ and $(x_0, y_0) \in \text{gr}F$. Then

- (i) $T_S^{r(m)}(x, 0_X, \dots, 0_X) = T_S^r(x)$;
- (ii) $D_R^m F(x_0, y_0, 0_X, 0_Y, \dots, 0_X, 0_Y) = D^r F(x_0, y_0)$.

Proposition 2.2 (see [13, Proposition 3.4]) Let $S = \text{dom}F$ and $(x_0, y_0) \in \text{gr}F$. Then, for all $x \in S$,

- (i) $F(x) - \{y_0\} \subseteq D^r F(x_0, y_0)(x - x_0)$;
- (ii) $F(x) - \{y_0\} \subseteq T_{F(S)}^r(y_0)$.

Take $x = x_0$ in Proposition 2.2, one obtains the following results.

Corollary 2.1 Let $S = \text{dom}F$ and $(x_0, y_0) \in \text{gr}F$. Then

- (i) $0_Y \in D^r F(x_0, y_0)(0_X)$;
- (ii) $0_Y \in T_{F(S)}^r(y_0)$.

In this paper, consider the following set-valued vector optimization problem:

$$(P) \begin{cases} \min & F(x), \\ \text{s.t.} & x \in S, G(x) \cap (-D) \neq \emptyset. \end{cases}$$

The point $(x_0, y_0) \in \text{gr}F$ is said to be a weakly efficient solution of problem (P) if

$$(F(A) - \{y_0\}) \cap (-\text{int}D) = \emptyset,$$

where $A := \{x \in S \mid G(x) \cap (-D) \neq \emptyset\}$.

For other notations and definitions, one refers to Ref. [13].

3 Properties of Higher-Order Upper Radial Sets and Upper Radial Derivatives

Proposition 3.1 Let $x \in S \subseteq X$, $F : S \rightarrow 2^Y$, $(x_0, y_0) \in \text{gr}F$ and $u_i = 0_X \in X, v_i \in -C, i = 1, \dots, m - 1$. Then

$$T_{F_+(S)}^{r(m)}(y_0, v_1, \dots, v_{m-1}) \subset T_{F_+(S)}^r(y_0), \tag{1}$$

and

$$D_R^m F_+(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x) \subset D^r F_+(x_0, y_0)(x), \forall x \in X. \tag{2}$$

Proof. We first prove that (1) holds.

Let $y \in T_{F_+(S)}^{r(m)}(y_0, v_1, \dots, v_{m-1})$. Then there exist sequences $t_n > 0, x_n \in S$ and $y_n \in F(x_n) + C$ such that

$$\frac{y_n - y_0 - t_n v_1 - \dots - t_n^{m-1} v_{m-1}}{t_n^m} \rightarrow y. \tag{3}$$

Since $y_n \in F(x_n) + C, v_i \in -C, i = 1, \dots, m - 1$ and $t_n > 0, \bar{y}_n := y_n - t_n v_1 - \dots - t_n^{m-1} v_{m-1} \in F(x_n) + C$. Combined with (16), we can conclude that

$$\frac{\bar{y}_n - y_0}{t_n^m} \rightarrow y,$$

which implies $y \in T_{F_+(S)}^r(y_0)$. Thus (1) holds.

we next prove that (2) holds. Let $x \in X$. Let us consider two possible cases for $D_R^m F_+(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x)$.

Case 1: $D_R^m F_+(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x) = \emptyset$. (2) holds trivially.

Case 2: $D_R^m F_+(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x) \neq \emptyset$. Let $y \in D_R^m F_+(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x)$. Then there exist sequences $t_n > 0$, $x_n \in S$ and $y_n \in F(x_n) + C$ such that

$$\frac{(x_n, y_n) - (x_0, y_0) - t_n(u_1, v_1) - \dots - t_n^{m-1}(u_{m-1}, v_{m-1})}{t_n^m} \rightarrow (x, y). \quad (4)$$

Set $\bar{x}_n := x_n - t_n u_1 - \dots - t_n^{m-1} u_{m-1}$. Since $y_n \in F(x_n) + C$, $u_i = 0_X, v_i \in -C, i = 1, \dots, m-1$ and $t_n > 0$,

$$\bar{y}_n := y_n - t_n v_1 - \dots - t_n^{m-1} v_{m-1} \in F(\bar{x}_n) + C.$$

Combined with (4), we can conclude that

$$\frac{(\bar{x}_n, \bar{y}_n) - (x_0, y_0)}{t_n^m} \rightarrow (x, y),$$

which implies $y \in D^r F_+(x_0, y_0)(x)$. Thus (2) holds and the proof is complete. \square

By Proposition 3.1 and [13, Remark 3.2(iv)], we have the following result.

Corollary 3.1 Let $x \in S \subseteq X$, $F : S \rightarrow 2^Y$, $(x_0, y_0) \in \text{gr}F$ and $u_i \in X, v_i \in -C, i = 1, \dots, m-1$. Then

$$D_R^m F_+(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(X) \subset T_{F_+(S)}^r(y_0).$$

Proposition 3.2 Let $x \in S \subseteq X$, $F : S \rightarrow 2^Y$, $(x_0, y_0) \in \text{gr}F$ and $u_i = 0_X, v_i \in C, i = 1, \dots, m-1$. Then

$$F(S) - \{y_0\} \subset T_{F_+(S)}^{r(m)}(y_0, v_1, \dots, v_{m-1}),$$

$$F(x) - \{y_0\} \subset D_R^m F_+(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x - x_0).$$

Proof. Let $x \in S$ and $y \in F(x)$. Take $x_n = x$, $y_n = y + v_1 + v_2 + \dots + v_{m-1}$ and $t_n = 1$. Since $v_i \in C, i = 1, \dots, m-1$, $y_n \in F(x_n) + C$. Thus, it follows from the definitions of m th-order upper radial sets that

$$\frac{(x_n, y_n) - (x_0, y_0) - t_n(u_1, v_1) - \dots - t_n^{m-1}(u_{m-1}, v_{m-1})}{t_n^m} = (x - x_0, y - y_0)$$

$$\in T_{\text{gr}F_+}^{r(m)}(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1}),$$

which implies

$$y - y_0 \in T_{F_+(S)}^{r(m)}(y_0, v_1, \dots, v_{m-1}),$$

$$y - y_0 \in D_R^m F_+(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x - x_0).$$

So $F(S) - \{y_0\} \subset T_{F_+(S)}^{r(m)}(y_0, v_1, \dots, v_{m-1})$ and $F(x) - \{y_0\} \subset D_R^m F_+(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x - x_0)$. The proof is complete. \square

4 Optimality Conditions

In this section, we present optimality conditions for weakly efficient solutions of set-valued optimization problems. In addition, we remove deficiencies contained in two earlier results in [13].

Theorem 4.1 Let $(x_0, y_0) \in \text{gr}F$ and $z_0 \in G(x_0) \cap (-D)$. If (x_0, y_0) is a weakly efficient pair of (P), then, the following separations holds

$$T_{(F,G)_+(S)}^r(y_0, z_0) \cap -\text{int}(C \times D) = \emptyset, \tag{5}$$

and

$$D^r(F, G)_+(x_0, y_0, z_0)(X) \cap -\text{int}(C \times D) = \emptyset. \tag{6}$$

Proof. By Proposition 2.1 and [13, Remark 3.2(iv)], we can easily derive (6) from (5). Therefore, we need to prove only that (5) holds. Suppose that (5) does not hold. Then, there exists $(y, z) \in Y \times Z$ such that

$$(y, z) \in T_{(F,G)_+(S)}^r(y_0, z_0) \tag{7}$$

and

$$(y, z) \in -\text{int}(C \times D). \tag{8}$$

It follows from (7) and the definition of first-order upper radial sets that there exist sequences $t_n > 0, x_n \in S$ and $(y_n, z_n) \in (F, G)(x_n) + C \times D$ such that

$$\frac{(y_n, z_n) - (y_0, z_0)}{t_n} \rightarrow (y, z). \tag{9}$$

From (8),(9) and $z_0 \in -D$, there exists large enough natural number N such that

$$y_n - y_0 \in -intC, z_n \in -intD, \forall n > N. \tag{10}$$

Since $z_n \in G(x_n) + D$, there exist $\bar{z}_n \in G(x_n)$ and $d_n \in D$ such that $z_n = \bar{z}_n + d_n$. It follows from (10) that $\bar{z}_n \in G(x_n) \cap (-D), \forall n > N$, which implies $x_n \in A$, for any $n > N$. Since $y_n \in F(x_n) + C$, there exist $\bar{y}_n \in F(x_n)$ and $c_n \in C$ such that $y_n = \bar{y}_n + c_n$. It follows from (10) that

$$\bar{y}_n - y_0 \in (F(x_n) - \{y_0\}) \cap (-intC) \subset (F(A) - \{y_0\}) \cap (-intC), \forall n > N,$$

which contradicts that (x_0, y_0) be a weakly efficient pair of (P). So (5) holds and the proof is complete. □

Remark 4.1 *By Proposition 3.1 and Corollary 3.1, we can easily derive [13, Theorem 4.1] from Theorem 4.1.*

Corollary 4.1 (see [13, Theorem 4.1]) Let $(x_0, y_0) \in \text{gr}F$ be a weakly efficient pair of (P), $z_0 \in G(x_0) \cap (-D)$, $(u_i, v_i, w_i) \in X \times (-C) \times (-D), i = 1, 2, \dots, m - 1$. Then, the following separations holds

$$T_{(F,G)_+(S)}^{r(m)}((y_0, z_0), (v_1, w_1), \dots, (v_{m-1}, w_{m-1})) \cap -int(C \times D) = \emptyset$$

and

$$D_R^m(F, G)_+(x_0, y_0, z_0, u_1, v_1, w_1, \dots, u_{m-1}, v_{m-1}, w_{m-1})(X) \cap -int(C \times D) = \emptyset.$$

Theorem 4.2 Let $(x_0, y_0) \in \text{gr}F$, $z_0 \in G(x_0) \cap (-D)$, $(u_i, v_i, w_i) \in \{0_X\} \times C \times D, i = 1, 2, \dots, m - 1$. If one of the following separations holds

(i)

$$T_{(F,G)_+(S)}^{r(m)}((y_0, z_0), (v_1, w_1), \dots, (v_{m-1}, w_{m-1})) \cap -(intC \times D(z_0)) = \emptyset, \tag{11}$$

(ii)

$$D_R^m(F, G)_+(x_0, y_0, z_0, u_1, v_1, w_1, \dots, u_{m-1}, v_{m-1}, w_{m-1})(x - x_0) \cap -(intC \times D(z_0)) = \emptyset, x \in A, \tag{12}$$

then (x_0, y_0) is a weakly efficient pair of (P).

Proof. We need to prove only that (i) holds. It follows from Proposition 3.2 that

$$(F, G)(x) - \{(y_0, z_0)\} \subset T_{(F,G)_+(S)}^{r(m)}((y_0, z_0), (v_1, w_1), \dots, (v_{m-1}, w_{m-1})), \forall x \in A.$$

Thus, by (11), we have

$$[(F, G)(x) - \{(y_0, z_0)\}] \cap -(intC \times D(z_0)) = \emptyset, \forall x \in A. \tag{13}$$

Suppose that there exist $x \in A$ and $y \in F(x)$ such that $y - y_0 \in -intC$. Then there exists $z \in G(x) \cap (-D)$ such that $z - z_0 \in -D(z_0)$, and hence

$$(y, z) - (y_0, z_0) \in -(intC \times D(z_0)),$$

which contradicts (13). So (x_0, y_0) is a weakly efficient pair of (P) and the proof is complete. □

Corollary 4.2 Let $(x_0, y_0) \in grF$ and $z_0 \in G(x_0) \cap (-D)$. If one of the following separations holds

- (i) $T_{(F,G)_+(S)}^r((y_0, z_0)) \cap -(intC \times D(z_0)) = \emptyset,$
- (ii) $D^r(F, G)_+(x_0, y_0, z_0)(x - x_0) \cap -(intC \times D(z_0)) = \emptyset, x \in A,$

then (x_0, y_0) is a weakly efficient pair of (P).

Remark 4.2 (i) Since [13, (8)] need be satisfied for any vector group $(u_i, v_i, w_i) \in X \times (-C) \times (-D), i = 1, 2, \dots, m - 1,$ and equalities (11) and (12) need be satisfied for a vector group $(u_i, v_i, w_i) \in \{0_X\} \times C \times D, i = 1, 2, \dots, m - 1,$ Theorem 4.2 improves [13, Theorem 4.4].

(ii) Take $G(x) \equiv Z$. Then [12, Theorem 4.4] can be obtained from Corollary 4.2 and Theorem 4.1.

(iii) By Corollary 4.2 and the proof of [13, Theorem 4.4], [13, Theorem 4.4] can be derived from Theorem 4.2.

Corollary 4.3 (see [13, Theorem 4.4]) Let $(x_0, y_0) \in grF$. Suppose that there exists $z_0 \in G(x_0) \cap (-D)$ such that, for $(u_i, v_i, w_i) \in X \times (-C) \times (-D), i = 1, 2, \dots, m - 1,$ and x in the feasible set $A,$ one of the following separations holds

- (i) $T_{(F,G)_+(S)}^{r(m)}((y_0, z_0), (v_1, w_1), \dots, (v_{m-1}, w_{m-1})) \cap -(intC \times D(z_0)) = \emptyset,$

(ii) $D_R^m(F, G)_+(x_0, y_0, z_0, u_1, v_1, w_1, \dots, u_{m-1}, v_{m-1}, w_{m-1})(x-x_0) \cap -(intC \times D(z_0)) = \emptyset$.

Then, (x_0, y_0) is a weakly efficient pair of (P).

By employing m th-order upper radial set and m th-order upper radial derivative, Anh et al.(see [13]) established the following sufficient optimality conditions of weakly efficient solutions for (P):

Theorem A (see [13, Theorem 4.5]) Let the assumptions of [13, Theorem 4.4] be satisfied. Then, (x_0, y_0) is a weakly efficient pair of (P) if one of the following conditions holds.

(i) For all $(y, z) \in T_{(F,G)_+}^{r(m)}(A)((y_0, z_0), (v_1, w_1), \dots, (v_{m-1}, w_{m-1}))$, there exists $(c^*, d^*) \in C^* \times D^* \setminus \{0, 0\}$ such that $\langle d^*, z_0 \rangle = 0$ and

$$\langle c^*, y \rangle + \langle d^*, z \rangle > 0. \quad (14)$$

(ii) For all $x \in A$ and all $(y, z) \in D_R^m(F, G)_+(x_0, y_0, z_0, u_1, v_1, w_1, \dots, u_{m-1}, v_{m-1}, w_{m-1})(x-x_0)$, there exists $(c^*, d^*) \in C^* \times D^* \setminus \{0, 0\}$ such that $\langle d^*, z_0 \rangle = 0$ and

$$\langle c^*, y \rangle + \langle d^*, z \rangle > 0. \quad (15)$$

Theorem B (see [13, Theorem 4.6]) For problem (P), $(x_0, y_0) \in \text{gr}F$, $z_0 \in G(x_0) \cap (-D)$. Let $(e, k) \in \text{int}(C \times D)$. Then, (x_0, y_0) is a weakly efficient pair of (P) if one of the following conditions holds.

(i) There exists $(\Gamma, L) \subset C^* \times D^* \setminus \{(0, 0)\}$ such that

$$C = \{y \in Y | \langle f, y \rangle \geq 0, \text{ for any } f \in \Gamma\}, D = \{z \in Z | \langle g, z \rangle \geq 0, \text{ for any } g \in L\},$$

$$\sup_{(f,g) \in (\Gamma, L)} \left\{ \frac{\langle f, 0_Y \rangle + \langle g, -z_0 \rangle}{\langle f, e \rangle + \langle g, k \rangle} \right\} = 0, \quad (16)$$

and

$$\sup_{(f,g) \in (\Gamma, L)} \left\{ \frac{\langle f, y \rangle + \langle g, z \rangle}{\langle f, e \rangle + \langle g, k \rangle} \right\} > 0 \quad (17)$$

for any $(y, z) \in T_{(F,G)_+(A)}^{r(m)}((y_0, z_0), (v_1, w_1), \dots, (v_{m-1}, w_{m-1}))$.

(ii) (16) and (17) satisfy for all $(y, z) \in D_R^m(F, G)_+(x_0, y_0, z_0, u_1, v_1, w_1, \dots, u_{m-1}, v_{m-1}, w_{m-1})(x-x_0)$ for each $x \in A$.

Unfortunately, [13, Theorems 4.5 and 4.6] exist gaps. On the one hand, from the proofs of [13, Theorems 4.4 and 4.6], it is possible that $u_i = 0_Y$ and $v_i = 0_Z, i = 1, 2, \dots, m - 1$ in [13, Theorems 4.5 and 4.6], and then, the assumptions of [13, Theorems 4.5 and 4.6] should be satisfied for the case. In fact, by Proposition 2.1 and Corollary 2.1, for any $(x, y) \in grF, z \in G(x)$, one obtains $(0_Y, 0_Z) \in T_{(F,G)_+(A)}^{r(m)}(y, z, 0_Y, 0_Z, \dots, 0_Y, 0_Z) = T_{(F,G)_+(A)}^r(y, z)$ and $(0_Y, 0_Z) \in D_R^m(F, G)_+(x, y, z, 0_X, 0_Y, 0_Z, \dots, 0_X, 0_Y, 0_Z)(0_X)$. Therefore, for any $(\Gamma, L) \subset (C^* \times D^*) \setminus (0_{Y^*}, 0_{Z^*})$, the conditions (14), (15) and (17) never hold. On the other hand, the condition (16) can be simply written as

$$z_0 = 0_Z. \tag{18}$$

Indeed, (18) \Rightarrow (16) is obvious. In what concerns the implication (16) \Rightarrow (18), it follows from $z_0 \in -D$ that $g(-z_0) \geq 0$, for all $g \in L \subset D^+$. Thus, if (16) holds, then for all $(f, g) \in \Gamma \times L$, we have

$$0 \leq \frac{\langle f, 0_Y \rangle + \langle g, -z_0 \rangle}{\langle f, e \rangle + \langle g, k \rangle} \leq \sup_{(f', g') \in \Gamma \times L} \left\{ \frac{\langle f', 0_Y \rangle + \langle g', -z_0 \rangle}{\langle f', e \rangle + \langle g', k \rangle} \right\} = 0,$$

which implies $g(-z_0) = 0$, for all $g \in L$. This means that $-z_0, z_0 \in \{x \in Z | g(x) \geq 0, \forall g \in L\} = D$. Since D is pointed, one concludes that $z_0 = 0_Z$.

We next give Theorems 4.3 and 4.4 which are appropriate modifications for deficiencies contained in [13, Theorems 4.5 and 4.6].

Theorem 4.3 Let the assumptions of [13, Theorem 4.4] be satisfied. Then, (x_0, y_0) is a weakly efficient pair of (P) if one of the following conditions holds.

- (i) For any $(y, z) \in (T_{(F,G)_+(A)}^{r(m)}((y_0, z_0), (v_1, w_1), \dots, (v_{m-1}, w_{m-1})) \setminus \{(0_Y, 0_Z)\})$, there exists $(c^*, d^*) \in C^* \times D^* \setminus \{(0_{Y^*}, 0_{Z^*})\}$ such that $\langle d^*, z_0 \rangle = 0$ and $\langle c^*, y \rangle + \langle d^*, z \rangle > 0$.
- (ii) For each $x \in A$ and all $(y, z) \in D_R^m(F, G)_+(x_0, y_0, z_0, u_1, v_1, w_1, \dots, u_{m-1}, v_{m-1}, w_{m-1})(x - x_0)$ with $(y, z) \neq (0_Y, 0_Z)$, there exists $(c^*, d^*) \in C^* \times D^* \setminus \{(0_{Y^*}, 0_{Z^*})\}$ such that $\langle d^*, z_0 \rangle = 0$ and $\langle c^*, y \rangle + \langle d^*, z \rangle > 0$.

Theorem 4.4 Let $(x_0, y_0) \in grF, z_0 \in G(x_0) \cap (-D)$ and $(e, k) \in int(C \times D)$. Then, (x_0, y_0) is a weakly efficient pair of (P) if one of the following conditions holds.

- (i) $z_0 = 0_Z$ and there exists $(\Gamma, L) \subset (C^* \times D^*) \setminus \{(0_{Y^*}, 0_{Z^*})\}$ such that $C = \{y \in Y | \langle f, y \rangle \geq 0, \text{ for any } f \in \Gamma\}, D = \{z \in Z | \langle g, z \rangle \geq 0, \text{ for any } g \in L\},$

$$\sup_{(f,g) \in (\Gamma,L)} \left\{ \frac{\langle f, y \rangle + \langle g, z \rangle}{\langle f, e \rangle + \langle g, k \rangle} \right\} > 0, \tag{19}$$

for any $(y, z) \in T_{(F,G)_+(A)}^{r(m)}((y_0, z_0), (v_1, w_1), \dots, (v_{m-1}, w_{m-1})) \setminus \{(0_Y, 0_Z)\}$.

- (ii) $z_0 = 0_Z$ and (19) satisfy for all $(y, z) \in D_R^m(F, G)_+(x_0, y_0, z_0, u_1, v_1, w_1, \dots, u_{m-1}, v_{m-1}, w_{m-1})(x - x_0) \setminus \{(0_Y, 0_Z)\}$, for each $x \in A$.

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